

Finance

CY Cergy-Paris University
Sébastien Laffitte



Last week

- The objective of the firm
- Analysis of Financial Accounts.
 - Ratio analysis.

Today

- Introduction to financial mathematics.
- Future Value, Present Value and Yield.
- Single payment securities and multiple payment securities.

The time value of money

The \$1.8 billion Powerball jackpot (December 2025)

- In December 2025, an American lottery ticket won a jackpot of **\$1.817 billion**.
- The winner could choose between:
 - **Option A:** Receive \$834.9 million today.
 - **Option B:** Receive \$1.817 billion in 30 payments: one today, followed by 29 annual payments increasing by 5%.
- Which option would you choose?
- More generally, how can we compare cash flows received at **different points in time**?

The time value of money

- A dollar today is worth more than a dollar in the future.
 - Individuals prefer it.
 - Inflation.
 - Uncertainty about the future.
- Finance is about valuing assets but for this we need to compare them adequately.
- We cannot compare values at different points in time.
 - How much is \$1 received in one year worth today?

The time value of money

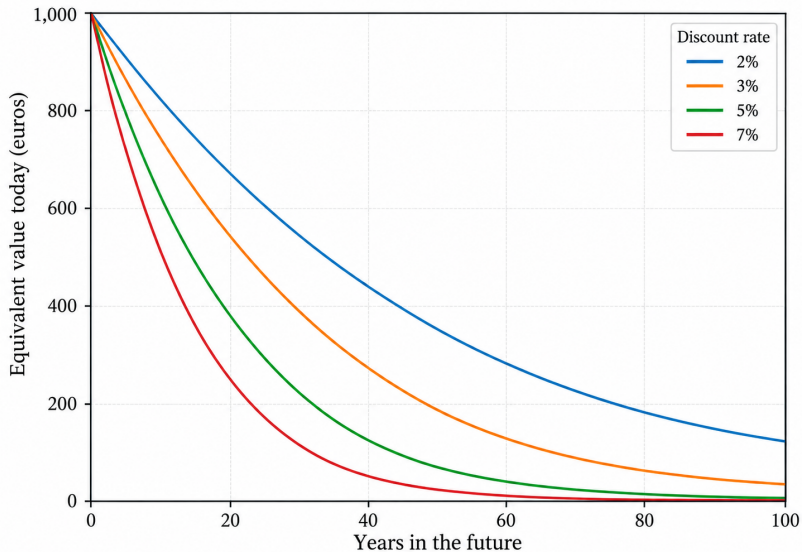
- **Discounting:** bring future CF into today's value.
 - **Compounding:** bring today's CF into the future.
 - The discount rate makes the link between each period. A higher discount rate means a higher preference for the present and then a lower value of the CF in the future.
 - Present value principles:
 - CF at different points in time cannot be aggregated. You need to bring them to the same point in time.
 - A good investment depends on how much CF you can receive, but also on when you receive it.
- This is the exchange rate between today's euros and tomorrow's euros.

The different types of Cash Flows

- Most assets are a combination of:
 - Simple CF: single CF in the future.
 - *e.g.* zero-coupon bond (*e.g.* U.S. T-bond) or security deposit.
 - Annuities: constant CF that occur at regular intervals (not necessarily annually).
 - *e.g.* standard bond or fixed-lease agreement
 - Growing annuity: a CF that grows over time (*e.g.* the CF increases by 5% each year).
 - *e.g.* bond indexed to inflation (U.S. TIPS), the Powerball example.
 - Perpetuity: constant CF at regular intervals forever.
 - *e.g.* preferred stock with fixed dividends
 - Growing perpetuity: growing CF forever.
 - *e.g.* common stock that pays dividends that increase each year.

The different types of Cash Flows

Present value of €1,000 received in the future

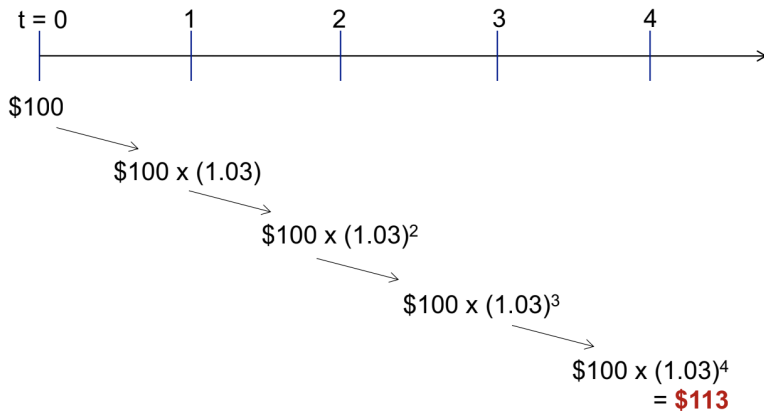


Future Value

- How much is \$1,000 invested for one year at an annual rate of 5%?
- $FV = PV \times (1 + r)^T = 1000 \times (1.05)^1 = 1050$
- How much is \$1,000 invested at an annual rate of 5% for two years?
- $FV = PV \times (1 + r)^T = 1000 \times (1.05)^2 = 1102.5$
- Can be decomposed as:

$$1102.5 = \underbrace{1000}_{\text{Principal}} + \underbrace{2 \times 0.05 \times 1000}_{\text{Simple interests}} + \underbrace{0.05 \times 0.05 \times 1000}_{\text{Interests on interests}}$$

Future Value



Present Value

- To receive \$1,000 in one year, how much should I invest today at a rate of 5%?
- $PV = \frac{FV}{(1+r)^t} = \frac{1000}{1+0.05} = 952.38$
- $\frac{1}{(1+r)^t}$ is the discount factor and r is the discount rate (or yield).
- To receive \$1,000 in two years, how much should I invest today at a rate of 5%?
- $PV = \frac{FV}{(1+r)^t} = \frac{1000}{(1+0.05)^2} = 907.03$

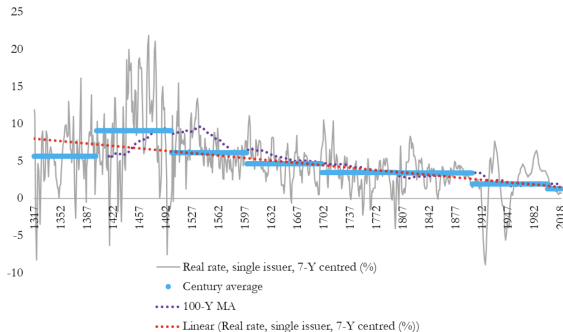
An example

- In 1626, a Dutch letter reported the “purchase” of Manhattan from Native Americans for 60 guilders. This is approximately \$24.
- In 2026, Manhattan’s real estate is valued at approximately \$545.2bn for tax purposes.
- Was it a good deal?
- Alternatively, suppose these \$24 had been invested in safe assets for 400 years.
 - Always think in terms of opportunity costs: what would have been the return to these \$24 if they were invested in a safe asset?
- What is the implied annual rate of return? Let’s compute the yield r .
- $PV = \frac{FV}{(1+r)^t} \Leftrightarrow r = \left(\frac{FV}{PV}\right)^{1/t} - 1$
- $r = \left(\frac{545.2 \times 10^9}{24}\right)^{\frac{1}{2026-1626}} - 1 \approx 6.14\%$

Sources: Dutch National Archives (Schaghen letter, 1626); NYC Department of Finance, FY2027 assessment roll. Historical comparison is illustrative.

An example

- The implied annualized return of Manhattan in our example is 6.14%.
- What is the historical return on safe assets?



Centennial averages

%	1300s	1400s	1500s	1600s	1700s	1800s	1900s	2000s
Nominal rate	7.3	11.2	7.8	5.4	4.1	3.5	5.0	3.5
Inflation	2.2	2.1	1.7	0.8	0.6	0.0	3.1	2.2
Real rate	5.1	9.1	6.1	4.6	3.5	3.4	2.0	1.3

Formulas

- $FV = PV(1 + r)^t$
- $PV = \frac{FV}{(1+r)^t}$
- $(1 + r) = \left(\frac{FV}{PV}\right)^{1/t}$
- $t = \frac{\log(\frac{FV}{PV})}{\log(1+r)}$

Exercise

- If you invested \$1,000 and received \$1,108 after 3 years, what was the interest rate?

- $(1 + r) = \left(\frac{FV}{PV}\right)^{1/t} = \left(\frac{1108}{1000}\right)^{1/3} = 1 + 0.0347$

- $r = 3.47\%$

- If you invested \$1,000 at 6% and received \$1,791, for how long did you invest?

- $t = \frac{\log(\frac{FV}{PV})}{\log(1+r)} = \frac{\log(\frac{1791}{1000})}{\log(1+0.06)} = 10$

- A clay tablet found in Babylon and kept in the Louvre poses the following problem: with an interest rate assumed to be 20%, how long will it take a loan to double in value at compound interest?

- $t = \frac{\log(\frac{FV}{PV})}{\log(1+r)} = \frac{\log(2)}{\log(1+0.2)} = 3.8 \text{ years or } 3 \text{ years and } 293 \text{ days.}$

Pricing securities: Single-payment security (Zero-coupon bond)

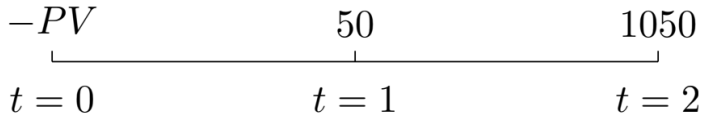
- The simplest fixed-income asset.
 - No interest payments.
 - At maturity, the owner is paid the par value of the bond (C).
 - The bond is bought below par value.
- The price is: $PV = \frac{C}{(1+r)^T}$
- Lower interest rates mean higher bond prices.

Pricing securities: Multiple-payment security

- Can be seen as a collection of zero-coupon bonds.
- PV of CF $C_0, C_1, \dots, C_T$:

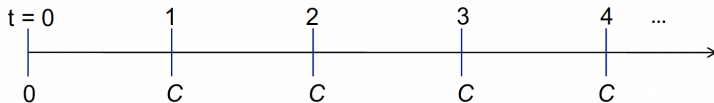
$$PV = C_0 + \frac{C_1}{(1+r_1)^1} + \dots + \frac{C_t}{(1+r_t)^t} + \dots + \frac{C_T}{(1+r_T)^T}$$

- What's the PV of this bond at a 5% interest rate?



Pricing securities: Perpetuities

- A perpetuity is a bond that pays a fixed payment C forever. No principal payment.
- Used in the 13th century between individuals or by states to raise capital.
- Existed in the U.K. and the U.S. more recently (see *consols*).
- Perpetuities help us think about stocks, which are infinite (residual) claims on corporate profits.

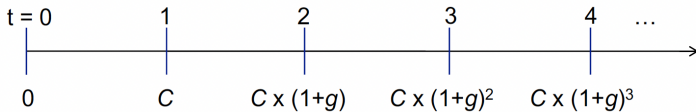


Pricing securities: Perpetuities

- $PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \dots = \frac{C}{r}$
- Note that the price decreases when the yield increases.
- Suppose your cell phone plan costs $20 \times 12 = 240$ euros a year. Assume the interest rate at all maturities is 5%.
- What price are you willing to pay for a "forever" plan?
- $PV = \frac{C}{r} = \frac{240}{0.05} = 4800.$

Pricing securities: Growing Perpetuities

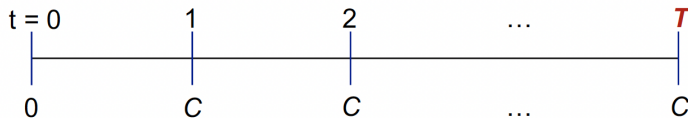
- A growing perpetuity is a stream of cash flows that occur at regular intervals and grow at a constant rate, g , forever.



- $$PV = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \dots = \frac{C}{r-g}$$

Pricing securities: Annuities

- An annuity is a stream of equal cash flows that occur at regular intervals for a finite number of T periods.



- $PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \dots + \frac{C}{(1+r)^T} = \sum_{i=1}^T \frac{C}{(1+r)^i} = \frac{C}{r} \left(1 - \frac{1}{(1+r)^T} \right)$
- We find C/r , just like in the perpetuity formula! We're just adjusting for the finite time horizon.

Pricing securities: Annuities

- What is the value of a 10%, 30-year mortgage that has an annual payment of 6,000 (starting next year)?
- $C = 6000, r = 0.1, T = 30. PV = \frac{6000}{0.1} \left(1 - \frac{1}{1.1^{30}}\right) = 56561,48$

Pricing securities: Annuities

- You started a new company with seed money of 18,000 from your parents and friends. You expect your company to earn 1,000 at the end of each year for the next 4 years (beginning in year 1), but earnings will then take off and grow by 7% per year forever. The discount rate is 10%.
- Is this new company a good idea?
- PV of first 4 years : $PV_1 = \frac{1000}{0.1} \left(1 - \frac{1}{(1+0.1)^4} \right) = 3169.87$
- PV of next years (5 to ∞) **at time 4**: $PV_2^{t=4} = \frac{1070}{0.1-0.07} = 35666$
- PV of next years (5 to ∞) **at time 0**: $PV_2^{t=0} = \frac{35666}{(1+0.1)^4} = 24360.81$
- $PV = PV_1 + PV_2^{t=0} = 3169.87 + 24360.81 = 27530.68$

Pricing security: the value of a government bond

- On October 1, 2026, France auctioned an OAT (10 years), with a coupon rate of 3.70%.
- For every 100 euros of face value, the bond pays 3.70 euros annually and 100 euros at maturity.
- The market price was 90.38 euros. What is the bond's yield to maturity (YTM)?
 - discount rate that equates the present value of all future payments to the current market price.
- Approximating the remaining maturity by 10 annual periods:

$$90.38 \approx \frac{3.70}{r} \left(1 - \frac{1}{(1+r)^{10}} \right) + \frac{100}{(1+r)^{10}}$$

- Solving for r gives $\boxed{YTM \approx 4.94\%}$.
- Why is the YTM higher than the coupon rate of 3.70%?
 - Investors pay less than 100€ but receive 100€ at maturity, in addition to the annual coupons. This is because 100€ in the future is worth less than 100€ today.

Pricing security: the value of a government bond

The failure of Silicon Valley Bank (2023)

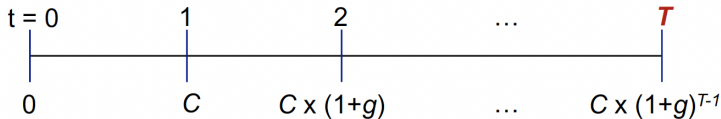
- In March 2023, Silicon Valley Bank failed following a bank run. Rising interest rates had reduced the market value of its long-term securities.
- At the end of 2022, its held-to-maturity securities had approximately \$15.2bn in unrealized losses.
- Recall that the price of a bond is the present value of its future cash flows: $P(r) = \sum_{t=1}^T \frac{CF_t}{(1+r)^t}$
- **Exercise:** Consider a 10-year zero-coupon bond that pays \$100 at maturity. What happens to its price if its yield rises from 1% to 4%?
- For a zero-coupon bond, $P(r) = \frac{F}{(1+r)^T}$:

$$P_{1\%} = \frac{100}{(1.01)^{10}} = 90.53; \quad P_{4\%} = \frac{100}{(1.04)^{10}} = 67.56$$

- The fall in bond values weakened SVB's balance sheet and contributed to the crisis.

Pricing securities: Growing Annuities

- A growing annuity is like a regular annuity, except that the cash flow C grows at a constant rate g .



- $$PV = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \dots + \frac{C(1+g)^{T-1}}{(1+r)^T} = \frac{C}{r-g} \left(1 - \frac{(1+g)^T}{(1+r)^T} \right)$$
- In all these formulas, CF begins in period 1 (one period after the present). If there is a CF in period 0, just add it separately.

Pricing securities: Growing Annuities

- In December 2025, the winner of the Powerball jackpot could choose between:
 - Option A: receive \$834.9 million today.
 - Option B: receive \$1.817 billion in 30 payments: one payment today, then 29 annual payments growing by 5%.
- At first sight, Option B looks much larger.
- But the payments do not occur at the same date.
- So the relevant question is:
 - What is the present value of Option B?

Pricing securities: Growing Annuities

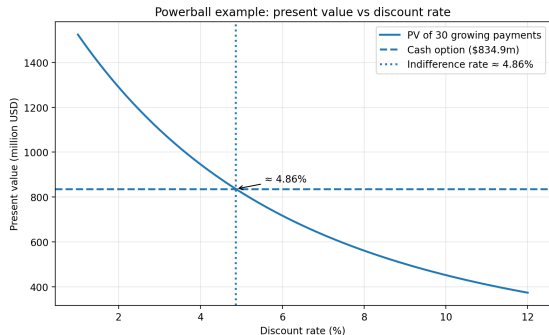
- Option A: \$834.9 million today.
- Option B: 30 growing payments, starting today.
- The first payment is \$27.35m. Using the growing annuity formula:

$$PV = C_0 + \frac{C_0(1+g)}{r-g} \left(1 - \frac{(1+g)^{29}}{(1+r)^{29}} \right)$$

- Suppose the discount rate is $r = 6\%$

$$PV = 27.35 + \frac{27.35(1.05)}{0.06 - 0.05} \left(1 - \frac{1.05^{29}}{1.06^{29}} \right)$$

$$PV \approx \$717.5\text{m}$$



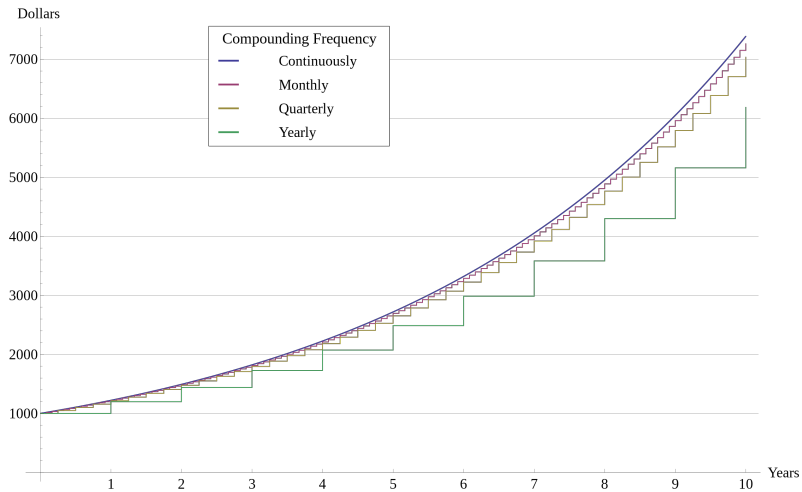
Calculating returns

- Here, we will look at different ways of compounding and quoting interest rates.
- The quoted rate is given on the basis customary in the market. We generally hear about yearly rates.
- However, rates can be compounded at different intervals. What if the yearly rate is compounded daily? Semi-annually?
- This will affect the future value differently. As a rule, any annual rate compounded more frequently will be worth more.
 - Annual quoted rate r with annual compounding means in t years: $FV = PV(1 + r)^t$
 - Annual quoted rate r with semi-annual compounding means in t years: $FV = PV(1 + r/2)^{2t}$
 - Annual quoted rate r with daily compounding means in t years: $FV = PV(1 + r/365)^{365t}$

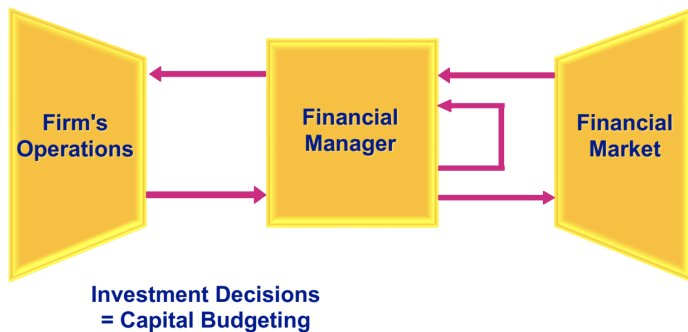
Continuous Compounding

- What happens to the FV when the rate is compounded continuously?
- Interesting theoretically and for professionals.
- $FV = PV \times e^{rt}$: simple formula.
- Suppose $r = 5\%$, $PV = 1000$, $t = 1$ year, what's the FV with annual compounding? With continuous compounding?
- Periodic: $FV = 1000 \times (1.05)^1 = 1050$.
- Continuous: $FV = 1000 \times e^{0.05} = 1051.27$.

Calculating returns



Investment decision rules



- How do firms decide what projects to invest in?
 - Facebook buys WhatsApp for \$19bn (2014).
 - Elon Musk buys Twitter for \$44bn (2022).
 - etc.

Investment decision rules

- We know how to compute the value of a project.
- Now, given the value of a project, is it worth investing in?
- We will study three decision tools:
 - The Net Present Value (NPV)
 - The Internal Rate of Return (IRR)
 - The Payback rule

The Net Present Value

- Reminder: The PV is the sum of the PV of individual CF:

$$PV = C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \dots + \frac{C_T}{(1+r)^T}$$

- The NPV is computed when there are some costs: investment costs for instance.
- The NPV corresponds to the difference between the PV of benefits and the PV of costs.
- NPV decision rule: Invest if and only if $NPV > 0$
- Careful: the NPV is sensitive to its assumptions: CF and r are uncertain.
 - Need to conduct a sensitivity analysis.
- First proposed as a systematic rule by Fischer (1907) but enters the theory of capital budgeting later in 1951.

The Net Present Value

Are investments in AI profitable?

- In 2026, tech companies are investing hundreds of billions of dollars in AI infrastructure.
- Suppose a company invests \$100bn today and expects additional cash flows of \$15bn per year for the next 10 years.
- Is this investment profitable?
- The NPV (in \$bn) is: $NPV = -100 + \frac{15}{1+r} + \dots + \frac{15}{(1+r)^{10}} = -100 + \frac{15}{r} \left(1 - \frac{1}{(1+r)^{10}}\right)$
- If $r = 8\%$: $NPV = +0.65\text{bn}$.
- If $r = 12\%$: $NPV = -15.25\text{bn}$.
- **An increase in the discount rate can turn a profitable investment into an unprofitable one!**

The Net Present Value

Parameter	Initial Assumption	Worst Case	Best Case
Units Sold (thousands)	50	35	65
Sale Price (\$/unit)	260	240	280
Cost of Goods (\$/unit)	110	120	100
NWC (\$ thousands)	1125	1525	725
Cost of Capital	12%	15%	10%

- Net working capital is a cash outflow ($\approx$ inventory)
- $CF = (Sale\ Price - COGS) * units - NWC$.
- NPV is

- Initial: $\frac{(260 - 110) * 50 - 1125}{1 + 0.12} = 5692$

- Worst: $\frac{(240 - 120) * 35 - 1525}{1 + 0.15} = 2326$

- Best: $\frac{(280 - 100) * 65 - 725}{1 + 0.10} = 9977$

The Internal Rate of Return

- Suppose you have the opportunity to pay 100 today and receive 120 in one year, knowing that similarly risky investments offer a return of 10%. Should you do it?
- We can directly see that the rate of return of this opportunity is 20%, which is larger than the rate of return of similar investments.
- It is less simple to compute the rate of return when there are multiple periods and CF.
- How to generalize this concept?

The Internal Rate of Return

- It is useful to see that the NPV is a function of the discount rate:

$$NPV(r) = C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \dots + \frac{C_T}{(1+r)^T}$$

- The IRR is the discount rate at which the NPV is exactly 0.
- It is the unknown in:

$$NPV(IRR) = C_0 + \frac{C_1}{1+IRR} + \frac{C_2}{(1+IRR)^2} + \dots + \frac{C_T}{(1+IRR)^T} = 0$$

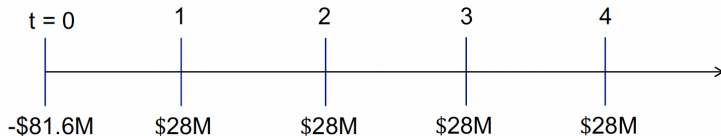
- IRR is not the cost of capital but the **hypothetical** cost of capital that would make the NPV zero. IRR depends on the CF while r depends on the risk of the project.

The Internal Rate of Return

- What is the IRR in the two-period case, with C_0 the initial investment (< 0) and C_1 the unique CF?
- $NPV(IRR) = -C_0 + \frac{C_1}{1+IRR} = 0$
- $\iff IRR = \frac{C_1 - C_0}{C_0}$
- $\longrightarrow$ Profit divided by investment.

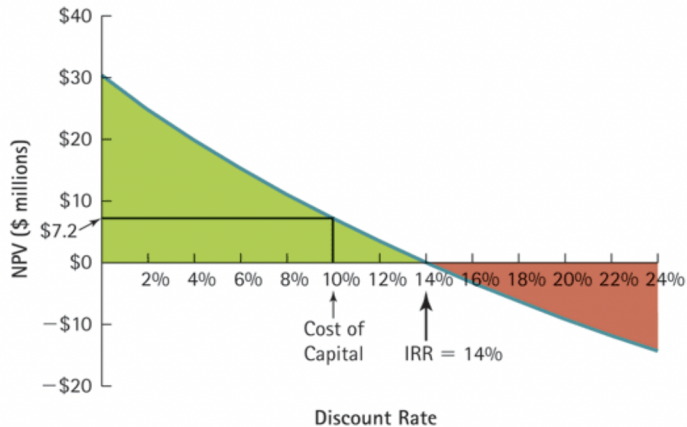
The Internal Rate of Return

- Generally, the IRR is not easy to determine.



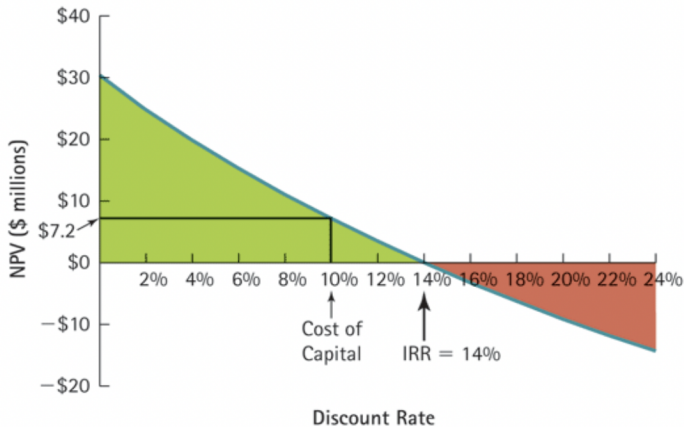
- By trial and error we can find it: compute $-81.6 + \frac{28}{R} \left(1 - \frac{1}{(1+R)^4} \right)$
- We obtain 0 for $R = 14$.

The Internal Rate of Return



The Internal Rate of Return

Naive Investment Rule



- Invest as long as the IRR is larger than the project's cost of capital (the return of other alternatives with equivalent risk and maturity).

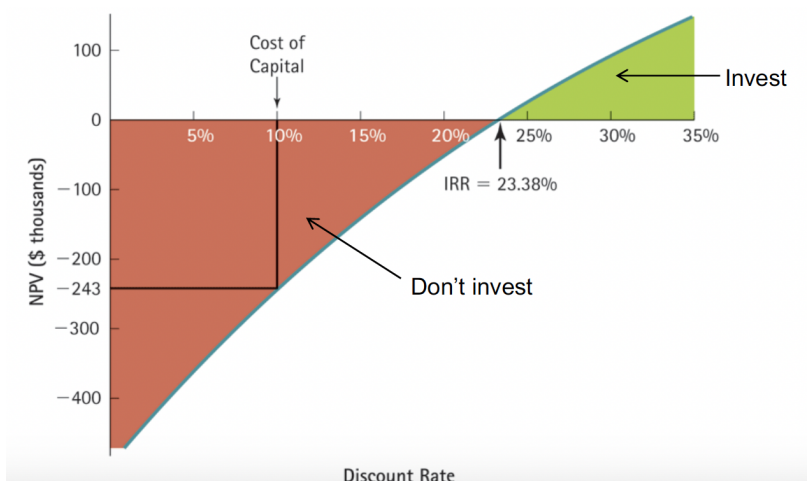
The Internal Rate of Return

Naive Investment Rule

- Why is it naive?
- It only works when the NPV is decreasing with the discount rate as in the case of investment projects.
 - When $C_0 < 0$ and next $CF > 0$.
- NPV can increase with the discount rate.
 - For debt for instance, where $C_0 > 0$ and other $CF < 0$.
- A less naive rule depends on the slope of $NPV(r)$.

The Internal Rate of Return

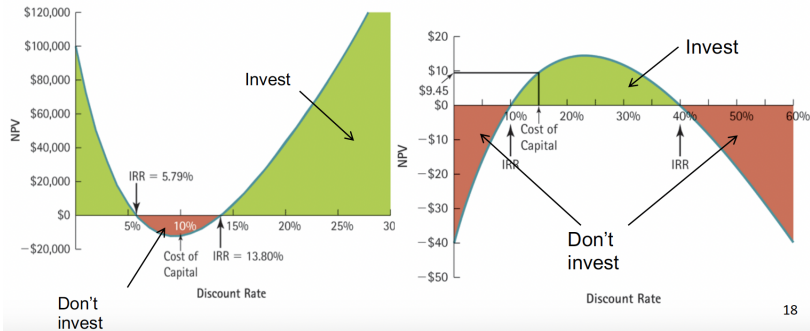
Naive Investment Rule



The Internal Rate of Return

Naive Investment Rule

- Still, it does not always work.
- $NPV(r)$ might not be a monotonic function.



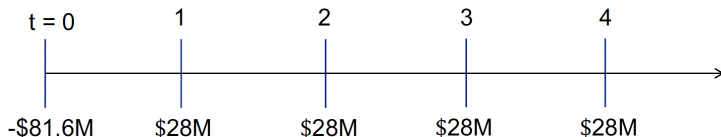
The Internal Rate of Return

IRR or NPV?

- For NPV, you need to compute only one number: the NPV at the cost of capital r .
- For IRR, you need to know the entire NPV function, which can be a complicated one.
- IRR can be relevant for projects with a simple CF structure.

Payback rule

- Payback is the amount of time it takes a project to pay back the initial investment.
- Rule: accept the project if the payback is lower than a pre-specified amount of time.



- What is the payback period?
- $28 \times 2 = 56 < 81.6 < 28 \times 3 = 84$. It's 3 years.

Payback rule

Drawbacks

- It ignores the time value of money. An upgrade would be to compute a discounted payback period.
- It ignores CF after the payback period: the payback rule can suggest to invest even if the NPV is negative.
- How should we choose the acceptable number of years?
- It is useful in case of an exogenous time constraint like a license to operate, a franchise agreement, etc.

Payback rule

Drawbacks

Project	C_0	C_1	C_2	C_3
A	-2,000	500	500	5,000
B	-2,000	500	1,800	0
C	-2,000	1,800	500	0

- Suppose the cutoff period is 2 years and $r = 10\%$
- Payback periods: 3, 2, 2: accept B and C
- NPV: 2624,-58,50: accept A and C

Choosing between projects in practice

What do CFOs do? Graham (2022)

- Share of firms that **always or almost always** use each investment decision rule:

	Large firms		Small firms	
	2001	2020	2001	2020
NPV	85%	77%	68%	40%
IRR	85%	75%	69%	40%
Payback	46%	64%	64%	66%

- NPV and IRR remain common in large firms, while smaller firms rely more on payback.
- Why do CFOs still use simple rules despite their limitations?

Choosing between projects in practice

What do CFOs do?

- NPV is the most informative and reliable rule.
- But it requires a lot of information:
 - Estimates of future CF.
 - Estimates of the cost of capital r .
 - Sensitivity analyses.
- Other rules can be simpler:
 - IRR doesn't require an estimate of r even if r is still needed to take the decision.
 - Payback rule does not require r nor CF far into the future (stop once the project is paid back).
- The practical decision must trade off accuracy and complexity.
- A good practice is to use several different rules.

Choosing between projects in practice

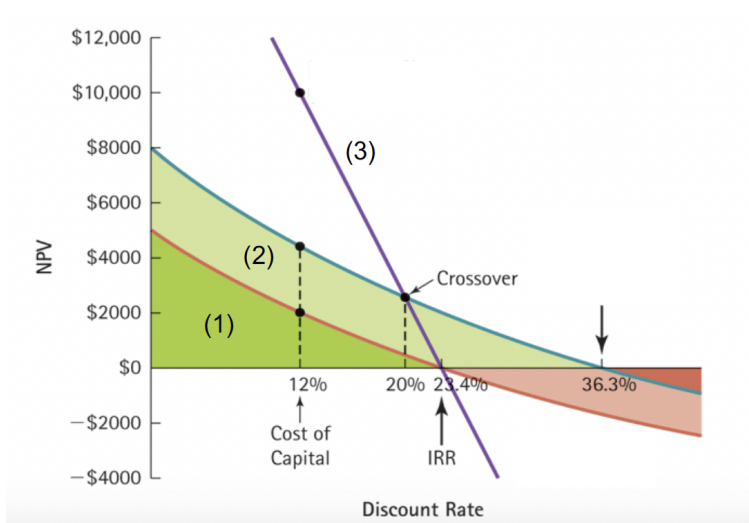
Choosing among mutually exclusive projects.

- Choose the project with the largest NPV
- It does not mean the project with the higher IRR!

	Investment	CF 1 st year	Growth rate (g)	Cost of capital (r)
1. Bar	\$400,000	\$60,000	3.5%	12%
2. Coffee Shop	\$200,000	\$40,000	3%	10%
3. Apparel Store	\$500,000	\$75,000	3%	13%

Choosing between projects in practice

Choosing among mutually exclusive projects.



Choosing between projects in practice

What if resources are limited?

- Suppose a firm has several projects with positive NPVs, but a limited investment budget.
- Which projects should it choose?
- Objective: Maximize total NPV subject to the budget constraint.
- A useful rule is to rank projects by their NPV per euro invested: $\frac{NPV_i}{I_i}$
- Example: A firm has a budget of 100m euros. Which project should it choose?
 - Project A: investment 60m, NPV 30m.
 - Projects B and C: investment 50m each, NPV 24m each.
- Choose B and C! Total NPV = 48, compared with 30 for project A. This is the case because projects are indivisible